# Building Database Management on top of Category Theory in Coq

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This document is available at http://web.mit.edu/jgross/Public/POPL/jgross-student-talk.pdf.
My category theory library is available at https://bitbucket.org/JasonGross/catdb.

# Outline

## Introduction — Databases and Category Theory

- Categories
- Relational Databases
- Relational Database Schema = Category
- Usefulness

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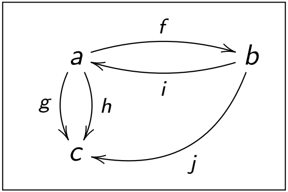

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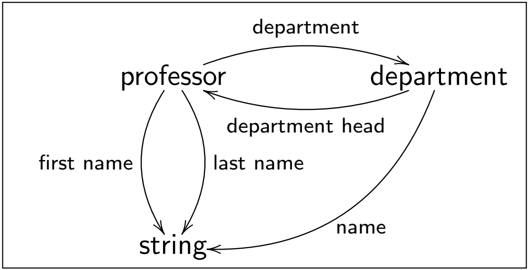

# Relational Database Schema = Category

$$
\begin{tikzcd}
a \arrow[r, bend left, "f"] \arrow[d, bend right, "g"'] \arrow[d, bend left, "h"] & b \arrow[l, bend left, "i"] \arrow[dl, bend left, "j"] \\
c &
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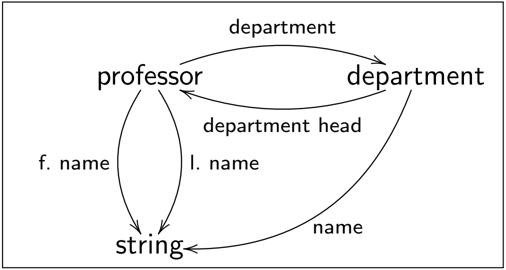

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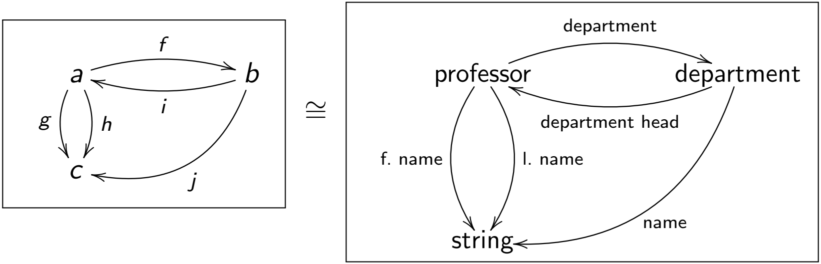


The diagrams are “the same”.

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- Provides a rigorous language for data migration between databases (another hard task in standard database management).

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- Category theory is relatively simple to code up.
  - Standard rigorous formulation of concepts exists in the literature.
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  - If you can define something categorically, it's probably interesting.

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- In some cases, Coq can infer the universe level of an inductive type from the universe levels of its parameters; when this happens, the inductive type is polymorphic over universe levels.
- It's useful to talk about "a category whose objects are of type `T`" rather than just "a category".

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- Categorical **colimits** are like **disjoint unions**, modulo equivalence relations

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  - Product types provide products (function types, e.g, `forall a : A, f a` is the product $\prod_{a \in A} f(a)$)
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- Coq has some colimits
  - Sigma types provide disjoint unions (e.g., `{ j : J & f j }` is the disjoint union $\coprod_{j \in J} f(j)$)
  - Quotients are . . . hard

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- Quotients can be defined via setoids
  - All objects carry around extra information of what the equivalence relation is
  - This is somewhat clunky
  - Not first-class quotients

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- There are two categorical constructions (limits and colimits) that are “dual”
- Coq’s type-system fully implements only one of these (limits)
- It’s harder to define colimits inside of Coq than limits, in general, even for the ones that Coq does support

# Thank You!